Math. Z. 252, 287–293 (2006) **Mathematische Zeitschrift**

DOI: 10.1007/s00209-005-0848-z

A monotonicity formula for stationary biharmonic maps*

Gilles Angelsberg

Departement für Mathematik, ETH Zentrum, Rämistrasse 101, CH-8092 Zürich, Switzerland (e-mail: gilles@math.ethz.ch)

Received: 29 October 2004; in final form: 7 February 2005 /
Published online: 16 August 2005 –

Abstract. We give a rigorous proof of the monotonicity formula of S.-Y.A. Chang, L. Wang and P. Yang [3] for (extrinsically) stationary biharmonic maps of class $W^{2,2}$.

1. Introduction

For several decades regularity properties of weakly harmonic maps have been intensely studied. For a manifold M of dimension $m \leq 2$, C.B. Morrey [9] showed in 1948 that every minimizing map $u \in W^{1,2}(M, N)$ belongs to $C^\infty(M, N)$. The regularity problem for general critical points of the harmonic energy functional had remained open for a long time. In 1981, still for the case $m \leq 2$, M. Grüter [5] proved smoothness of conformal weakly harmonic maps. R. Schoen [12] introduced the notion of stationary harmonic maps and extended Grüter's result to this class. Finally, F. Hélein [6] showed that every weakly harmonic map in the case $m \leq 2$ is smooth. For $m \geq 3$, more complex phenomena show up. R. Schoen and K. Uhlenbeck [13] showed that if $u \in W^{1,2}(M, N)$ is energy minimizing, then u is smooth except on a closed subset of finite H^{m-3}-measure. This result is optimal since the radial projection from B^m into S^{m-1} is a minimizing map for $m \geq 3$, as shown by H. Brézis, J.-M. Coron and E. Lieb [2] for $m = 3$ and F.H. Lin [8] for $m \geq 3$. On the other hand, T. Rivière [11] proved existence of everywhere discontinuous weakly harmonic maps. For the intermediate class of stationary harmonic maps, L.C. Evans [4] showed partial regularity for maps into the sphere and F. Bethuel [1] generalized this result for arbitrary target manifolds. Their proofs rely on a monotonicity formula for stationary harmonic maps adapted from P. Price [10].

* This work was partially supported by SNF 200021-101930/1.

Similar questions for (extrinsically) weakly biharmonic maps were studied by S.-Y.A. Chang, L. Wang and P. Yang in [3]. They showed smoothness for weakly biharmonic maps into the sphere and $m \leq 4$, and asserted partial regularity for stationary biharmonic maps into the sphere and $m \geq 5$. C.Y. Wang generalized these results for arbitrary target manifolds in [14] and [15]. Once again, a monotonicity formula derived from the stationarity assumption is crucial in the proof of partial regularity for $m \geq 5$. However, the derivation in [3, Proposition 3.2] is only given for sufficiently regular maps.

Here, we give a rigorous proof of this monotonicity formula in the case of stationary biharmonic maps of class $W^{2,2}(B_r, N)$.

Conceivably, this monotonicity formula may allow to study the singular behaviour of stationary biharmonic maps and especially minimizing biharmonic maps as suggested by M-C. Hong and C.Y. Wang in [7].

2. Stationarity Assumption

We recall some definitions and fix the notations used henceforth: Let B_r be the open ball of radius $r > 0$ in $\mathbb{R}^m$ centered at the origin and let N be a smooth Riemannian manifold isometrically embedded in $\mathbb{R}^n$. Recall that $W^{2,2}(B_r, N)$ is defined by

$$W^{2,2}(B_r, N) := \{u \in W^{2,2}(B_r, \mathbb{R}^n) : u(x) \in N \text{ for a.e. } x \in B_r\}.$$

We consider on $W^{2,2}(B_r, N)$ the Hessian energy functional

$$E(u) := \int_{B_r} |\Delta u|^2 dx.$$

Definition 1. *A map u in $W^{2,2}(B_r, N)$ is called (extrinsically) weakly biharmonic if it is a critical point of the Hessian energy $E(\cdot)$ with respect to compactly supported variations on the target manifold, i.e. if*

$$\frac{d}{dt}\,|_{t=0}\; E\left(\pi \circ (u + t\xi)\right) = 0 \textit{ for all } \xi \in C_0^\infty(B_r, \mathbb{R}^n),$$

where π denotes the nearest point projection onto N.

Definition 2. *A weakly biharmonic map u in $W^{2,2}(B_r, N)$ is called (extrinsically) stationary biharmonic if, in addition, u is a critical point of the Hessian energy $E(\cdot)$ with respect to compactly supported variations on the domain manifold, i.e. if*

$$\frac{d}{dt}\,|_{t=0}\; E\left(u \circ (id + t\xi)\right) = 0 \textit{ for all } \xi \in C_0^\infty(B_r, \mathbb{R}^m), \tag{1}$$

where id denotes the identity map.

In what follows we denote $u = (u^1, \ldots, u^n)$, $u_j = \frac{\partial u}{\partial x_j}$ etc. We tacitly sum over repeated indices. We use the following notation

$$\int_{\partial B_r \setminus \partial B_\rho} f d\sigma := \int_{\partial B_r} f d\sigma - \int_{\partial B_\rho} f d\sigma.$$

We replace Lemma 3.1 in [3] by the following lemma.

Lemma 1. *If u is a stationary biharmonic map in $W^{2,2}(B_{2r}, N)$, then we have:*

$$\int_{B_{2r}} \left(4u_{kk}u_{ij}\xi_i^j + 2u_{kk}u_j\xi_{ii}^j - |\Delta u|^2 \nabla \cdot \xi\right) dx = 0$$

for every test function $\xi \in \mathcal{C}_0^\infty(B_{2r}, \mathbb{R}^m)$.

Proof. We compute

$$\Delta u_t = (\Delta u)_t + t\left(2(u_{ij})_t\xi_i^j + (u_j)_t\xi_{ii}^j\right) + t^2(u_{jk})_t\xi_i^k\xi_i^j,$$

where for any f we denote $f_t(y) := f(y + t\xi(y))$ for $\xi \in \mathcal{C}_0^\infty(B_{2r}, \mathbb{R}^m)$. The change of variables $x = y + t\xi$ for $|t|$ sufficiently small induces a $\mathcal{C}^\infty$-diffeomorphism of B_{2r} onto itself. The stationarity assumption (1) then implies:

$$\begin{aligned} 0 &= \frac{d}{dt}\,|_{t=0} \int_{B_{2r}} |\Delta u_t|^2(x - t\xi)\left(1 - t\nabla \cdot \xi + O(t^2)\right) dx \\ &= \int_{B_{2r}} \left(4u_{kk}u_{ij}\xi_i^j + 2u_{kk}u_j\xi_{ii}^j - |\Delta u|^2 \nabla \cdot \xi\right) dx. \end{aligned}$$

□

3. Monotonicity formula

We can now rigorously derive the monotonicity formula Proposition 3.2. in [3] for almost every radius. This is sufficient to prove partial regularity for (extrinsically) stationary biharmonic maps of class $W^{2,2}$.

Theorem 1 (Monotonicity formula). *For $K > 0$ and $u \in W^{2,2}(B_K, N)$ (extrinsically) stationary biharmonic, it holds for a.e. $0 < \rho < r \leq \frac{K}{2}$*

$$r^{4-m}\int_{B_r} |\Delta u|^2 dx - \rho^{4-m}\int_{B_\rho} |\Delta u|^2 dx = P + R,$$

where

$$\begin{aligned} P &= 4\int_{B_r \setminus B_\rho} \left(\frac{(u_j + x^i u_{ij})^2}{|x|^{m-2}} + \frac{(m-2)(x^i u_i)^2}{|x|^m}\right) dx \\ R &= 2\int_{\partial B_r \setminus \partial B_\rho} \left(-\frac{x^i u_j u_{ij}}{|x|^{m-3}} + 2\frac{(x^i u_i)^2}{|x|^{m-1}} - 2\frac{|\nabla u|^2}{|x|^{m-3}}\right) d\sigma. \end{aligned}$$

Proof. Choose in Lemma 1 the test functions $\xi(x) := \psi^\epsilon\left(\frac{|x|}{\tau}\right)x$, where $\psi = \psi^\epsilon$: $\mathbb{R}_+ \to [0, 1]$ is smooth with compact support on $[0, 1]$ and $\psi^\epsilon \equiv 1$ on $[0, 1 - \epsilon]$. Then, it follows with $\xi_i^j = \psi_i x^j + \psi\delta_{ij}$ and $\xi_{ii}^j = \psi_{ii}x^j + 2\psi_j$

$$\begin{aligned} 0 = \int_{\mathbb{R}^m} \Big((4 - m)|\Delta u|^2\psi - |\Delta u|^2\psi_i x^i + 4u_{kk}u_{ij}\psi_i x^j \\ +4u_{kk}u_j\psi_j + 2u_{kk}u_j\psi_{ii}x^j\Big) dx. \end{aligned}$$

We have $\psi_i\left(\frac{|x|}{\tau}\right) = \frac{1}{\tau}\psi'\left(\frac{|x|}{\tau}\right)\frac{x^i}{|x|}$ and $\psi_{ii}\left(\frac{|x|}{\tau}\right) = \frac{1}{\tau^2}\psi''\left(\frac{|x|}{\tau}\right) + \frac{(m-1)}{\tau}\psi'\left(\frac{|x|}{\tau}\right)\frac{1}{|x|}$.

Thus,

$$0 = (4-m)\int_{\mathbb{R}^m} |\Delta u|^2 \psi dx - \frac{1}{\tau}\int_{\mathbb{R}^m} |\Delta u|^2 \psi' |x| dx + \frac{4}{\tau}\int_{\mathbb{R}^m} u_{kk} u_{ij} \psi' \frac{x^i x^j}{|x|} dx$$
$$+ \frac{2(m+1)}{\tau}\int_{\mathbb{R}^m} u_{kk} u_j \psi' \frac{x^j}{|x|} dx + \frac{2}{\tau^2}\int_{\mathbb{R}^m} u_{kk} u_j \psi'' x^j dx.$$

This implies for $I^\epsilon(\tau) := \tau^{4-m} \int_{\mathbb{R}^m} |\Delta u|^2 \psi^\epsilon \left(\frac{|x|}{\tau}\right) dx$

$$\begin{aligned}
\tau^{m-3}\frac{d}{d\tau} I^\epsilon(\tau) &= (4-m)\int_{\mathbb{R}^m} |\Delta u|^2 \psi dx - \frac{1}{\tau}\int_{\mathbb{R}^m} |\Delta u|^2 \psi' |x| dx \\
&= -\frac{4}{\tau}\int_{\mathbb{R}^m} u_{kk} u_{ij} \psi' \frac{x^i x^j}{|x|} dx - \frac{2(m+1)}{\tau}\int_{\mathbb{R}^m} u_{kk} u_j \psi' \frac{x^j}{|x|} dx \\
&\quad -\frac{2}{\tau^2}\int_{\mathbb{R}^m} u_{kk} u_j \psi'' x^j dx. \qquad (2)
\end{aligned}$$

Furthermore, we have:

$$\begin{aligned}
\int_\rho^r \psi''\left(\frac{|x|}{\tau}\right)\tau^{1-m} d\tau &= -\int_\rho^r \frac{d}{d\tau}\left(\psi'\left(\frac{|x|}{\tau}\right)\right)\tau^{3-m}\frac{1}{|x|} d\tau \\
&= -\psi'\left(\frac{|x|}{r}\right)\frac{r^{3-m}}{|x|} + \psi'\left(\frac{|x|}{\rho}\right)\frac{\rho^{3-m}}{|x|} \\
&\quad + \frac{3-m}{|x|}\int_\rho^r \psi'\left(\frac{|x|}{\tau}\right)\tau^{2-m} d\tau.
\end{aligned}$$

Thus, applying Fubini's theorem twice gives:

$$\begin{aligned}
&\int_\rho^r \tau^{1-m}\int_{\mathbb{R}^m} u_{kk} u_j \psi''\left(\frac{|x|}{\tau}\right) x^j dx\, d\tau \\
&= -\int_{\mathbb{R}^m} u_{kk} u_j x^j \psi'\left(\frac{|x|}{r}\right)\frac{r^{3-m}}{|x|} dx + \int_{\mathbb{R}^m} u_{kk} u_j x^j \psi'\left(\frac{|x|}{\rho}\right)\frac{\rho^{3-m}}{|x|} dx \\
&\quad +(3-m)\int_\rho^r \tau^{2-m}\int_{\mathbb{R}^m} u_{kk} u_j x^j \psi'\left(\frac{|x|}{\tau}\right)\frac{1}{|x|} dx\, d\tau. \qquad (3)
\end{aligned}$$

Multiplying both sides of (2) with τ^{3-m}, integrating over τ from ρ to r and inserting (3) yield

$$\begin{aligned}
I^\epsilon(r) - I^\epsilon(\rho) &= -4\int_\rho^r \tau^{2-m}\int_{\mathbb{R}^m} u_{kk} u_{ij} \psi'\left(\frac{|x|}{\tau}\right)\frac{x^i x^j}{|x|} dx\, d\tau \\
&\quad -8\int_\rho^r \tau^{2-m}\int_{\mathbb{R}^m} u_{kk} u_j \psi'\left(\frac{|x|}{\tau}\right)\frac{x^j}{|x|} dx\, d\tau \\
&\quad +2\int_{\mathbb{R}^m} u_{kk} u_j x^j \psi'\left(\frac{|x|}{r}\right) r^{3-m}\frac{1}{|x|} dx \\
&\quad -2\int_{\mathbb{R}^m} u_{kk} u_j x^j \psi'\left(\frac{|x|}{\rho}\right)\rho^{3-m}\frac{1}{|x|} dx.
\end{aligned}$$

For all Lebesgue points ρ and r of the function $g(s) := \int_{\partial B_s} \frac{u_{kk} u_j x^j}{|x|^{m-3}} d\sigma \in L^1_{\mathrm{loc}}(]0, K])$, as $\epsilon \to 0$, by applying Lemma 2 in the Appendix to the first two terms and the Lebesgue differentiation theorem to the last two terms, we obtain

$$
\begin{aligned}
&r^{4-m}\int_{B_r}|\Delta u|^2dx-\rho^{4-m}\int_{B_\rho}|\Delta u|^2dx\\
&=\int_{B_r\setminus B_\rho}\left(4\frac{u_{kk}u_{ij}x^ix^j}{|x|^{m-2}}+8\frac{u_{kk}u_jx^j}{|x|^{m-2}}\right)dx-2\int_{\partial B_r\setminus\partial B_\rho}\frac{u_{kk}u_jx^j}{|x|^{m-3}}d\sigma\\
&=2\int_{B_r\setminus B_\rho}\left(\frac{u_{kk}u_{ij}x^ix^j}{|x|^{m-2}}-\frac{u_{kki}u_jx^ix^j}{|x|^{m-2}}+\frac{u_{kk}u_jx^j}{|x|^{m-2}}\right)dx.
\end{aligned}
$$

Here and henceforth, all third derivatives are interpreted in the sense of distributions. The monotonicity formula now follows by several integrations by parts. We proceed as in [3] and compute for a.e. ρ and r

$$
\begin{aligned}
\int_{B_r\setminus B_\rho}\frac{u_{kk}u_{ij}x^ix^j}{|x|^{m-2}}dx&=\int_{\partial B_r\setminus\partial B_\rho}\frac{u_ku_{ij}x^ix^jx^k}{|x|^{m-1}}d\sigma-\int_{B_r\setminus B_\rho}\frac{u_ku_{ijk}x^ix^j}{|x|^{m-2}}dx\\
&\quad+\int_{B_r\setminus B_\rho}\left(-2\frac{u_ku_{ik}x^i}{|x|^{m-2}}+\frac{(m-2)u_ku_{ij}x^ix^jx^k}{|x|^m}\right)dx\\
&=\int_{\partial B_r\setminus\partial B_\rho}\left(\frac{u_ku_{ij}x^ix^jx^k}{|x|^{m-1}}-\frac{u_ku_{ik}x^i}{|x|^{m-3}}\right)d\sigma\\
&\quad+\int_{B_r\setminus B_\rho}\left(\frac{u_ku_{ik}x^i}{|x|^{m-2}}+\frac{u_{ik}u_{jk}x^ix^j}{|x|^{m-2}}\right)dx\\
&\quad+(m-2)\int_{B_r\setminus B_\rho}\frac{u_ku_{ij}x^ix^jx^k}{|x|^m}dx,\\
\int_{B_r\setminus B_\rho}-\frac{u_{kki}u_jx^ix^j}{|x|^{m-2}}dx&=-\int_{\partial B_r\setminus\partial B_\rho}\frac{u_{ki}u_jx^ix^jx^k}{|x|^{m-1}}\\
&\quad+\int_{B_r\setminus B_\rho}\left(\frac{u_{kj}u_jx^k}{|x|^{m-2}}+\frac{u_ju_{kk}x^j}{|x|^{m-2}}\right)dx\\
&\quad+\int_{B_r\setminus B_\rho}\left(\frac{(2-m)u_ju_{ik}x^ix^jx^k}{|x|^m}+\frac{u_{ik}u_{jk}x^ix^j}{|x|^{m-2}}\right)dx,\\
\int_{B_r\setminus B_\rho}\frac{u_{kk}u_jx^j}{|x|^{m-2}}dx&=\int_{\partial B_r\setminus\partial B_\rho}\frac{(u_kx^k)^2}{|x|^{m-1}}d\sigma-\int_{B_r\setminus B_\rho}\frac{|\nabla u|^2}{|x|^{m-2}}dx\\
&\quad+\int_{B_r\setminus B_\rho}\left(\frac{(m-2)(u_kx^k)^2}{|x|^m}-\frac{u_ku_{jk}x^j}{|x|^{m-2}}\right)dx.
\end{aligned}
$$

Combining these equations leads to

$$
\begin{aligned}
&r^{4-m}\int_{B_r}|\Delta u|^2dx-\rho^{4-m}\int_{B_\rho}|\Delta u|^2dx\\
&=2\int_{\partial B_r\setminus\partial B_\rho}\left(\frac{(u_ix^i)^2}{|x|^{m-1}}-\frac{u_ju_{ij}x^i}{|x|^{m-3}}\right)d\sigma\\
&\quad+2\int_{B_r\setminus B_\rho}\left(\frac{u_ju_{ij}x^i}{|x|^{m-2}}+2\frac{(u_{ij}x^i)^2}{|x|^{m-2}}+\frac{u_iu_{jj}x^i}{|x|^{m-2}}-\frac{|\nabla u|^2}{|x|^{m-2}}+\frac{(m-2)(u_ix^i)^2}{|x|^m}\right)dx.
\end{aligned}
$$

Moreover, it holds

$$0=-4\int_{\partial B_r\setminus\partial B_\rho}\frac{|\nabla u|^2}{|x|^{m-3}}d\sigma+8\int_{B_r\setminus B_\rho}\left(\frac{|\nabla u|^2}{|x|^{m-2}}+\frac{u_ju_{ij}x^i}{|x|^{m-2}}\right)dx$$

and

$$\begin{aligned}0=&\,2\int_{\partial B_r\setminus\partial B_\rho}\frac{(u_ix^i)^2}{|x|^{m-1}}d\sigma\\&-2\int_{B_r\setminus B_\rho}\left(\frac{u_ju_{ij}x^i}{|x|^{m-2}}+\frac{|\nabla u|^2}{|x|^{m-2}}+\frac{u_iu_{jj}x^i}{|x|^{m-2}}+\frac{(2-m)(u_ix^i)^2}{|x|^m}\right)dx.\end{aligned}$$

Adding the last three equations establishes the monotonicity formula. □

Appendix

For $f,\phi\in L^1(\mathbb{R},\mathbb{R})$, define

$$(\phi*f)(x):=\int_{\mathbb{R}}\phi\left(\frac{y}{x}\right)f(y)dy=\int_{\mathbb{R}}\phi(z)\,f(xz)\,xdz.$$

Then, there holds the following

Lemma 2. *Suppose* $f,(\phi_k)_{k\in\mathbb{N}}\in L^1(\mathbb{R},\mathbb{R})$*, with*

$$\phi_k\geq 0,\ \int_{\mathbb{R}}\phi_k=1\ \textit{and}\ \operatorname{supp}\phi_k\subset\left[1-\frac{1}{k},1\right],\ k\in\mathbb{N}.$$

Then, as $k\to\infty$ *for* $0<\rho<r$

$$\phi_k*f\to id\cdot f\ in\ L^1([\rho,r]),$$

where $(id\cdot f)(x)=xf(x)$.

Proof. Observe that

$$\|f(h\cdot)-f\|_{L^1(\mathbb{R},\mathbb{R})}\to 0\text{ for }h\to 1\tag{4}$$

and

$$(\phi_k*f)(x)-xf(x)=x\int_{\mathbb{R}}\phi_k(z)\,(f(xz)-f(x))\,dz.$$

Thus, we have using Fubini's theorem and Hölder's inequality:

$$\begin{aligned}\|\phi_k*f-xf\|_{L^1([\rho,r])}&\leq\int_\rho^r|x|\int_{\mathbb{R}}|\phi_k(z)||f(xz)-f(x)|dzdx\\&\leq r\int_{\mathbb{R}}|\phi_k(z)|\left(\int_\rho^r|f(xz)-f(x)|dx\right)dz\\&\leq r\|\phi_k\|_{L^1}\sup_{h\in\operatorname{supp}\phi_k}\|f(h\cdot)-f\|_{L^1}.\end{aligned}$$

As $\operatorname{supp}\phi_k\subset[1-\frac{1}{k},1]$ and $\|\phi_k\|_{L^1}\leq 1$, assumption (4) implies that the right hand side converges to zero for $k\to\infty$. □

Acknowledgements. I would like to thank Michael Struwe for his many helpful discussions.

References

1. Bethuel, F.: On the singular set of stationary harmonic maps. Manuscripta Math. **78**, 417–443 (1993)
2. Brézis, H., Coron, J.-M., Lieb, E.: Harmonic maps with defects. Comm. Math. Phys. **107**, 649–705 (1986)
3. Chang, S.-Y.A., Wang, L., Yang, P.: A regularity theory for biharmonic maps. Comm. Pure Appl. Math. **52**, 1113–1137 (1999)
4. Evans, L.C.: Partial regularity for stationary harmonic maps into the sphere. Arch. Rational Mech. Anal. **116**, 101–113 (1991)
5. Grüter, M.: Regularity of weak H-surfaces. J. reine angew. Math. **329**, 1–15 (1981)
6. Hélein, F.: Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne. C.R. Acad. Sci. Paris Sér. I Math. **312**, 591–596 (1991)
7. Hong, M.-C., Wang, C.Y.: Regularity and relaxed problems of minimizing biharmonic maps into spheres. Preprint
8. Lin, F.H.: Une remarque sur l'application $x/|x|$. C.R. Acad. Sci. Paris Sér. I Math. **305**, 529–531 (1987)
9. Morrey C.B. Jr.: The problem of Plateau on a Riemannian manifold. Ann. Math. **49**, 807–951 (1948)
10. Price, P.: A monotonicity formula for Yang-Mills fields. Manuscripta Math. **43**, 131–166 (1983)
11. Rivière, T.: Everywhere discontinuous harmonic maps into spheres. Acta Math. **175**, 197–226 (1995)
12. Schoen, R.: Analytic aspects of the harmonic map problem. Seminar on Nonlinear Partial Differential Equations (S.S. Chern editor), MSRI Publications, vol. 2, Springer, New York, 1984
13. Schoen, R., Uhlenbeck, K.: A regularity theory for harmonic maps. J. Differential Geometry **17**, 307–335 (1982)
14. Wang, C.Y.: Biharmonic maps from $\mathbb{R}^4$ into a Riemannian manifold. Math. Z. (to appear)
15. Wang, C.Y.: Stationary biharmonic maps from $\mathbb{R}^m$ into a Riemannian manifold. Comm. Pure Appl. Math. **57**, 419–444 (2004)